

Arithmetic of 3-valent graphs and equidissections of flat surfaces

October 20, 2015

Abstract

Our main object of study is a 3-valent graph with a vector function on its edges. The function assigns to each edge a pair of 2-adic integer numbers and satisfies additional condition: the sum of its values on the three edges, terminating in the same vertex, is equal to 0. For each vertex of the graph three vectors corresponding to these edges generate a lattice over the ring of 2-adic integers. In this paper we study the restrictions imposed on these lattices by the combinatorics of the graph.

As an application we obtain the following fact: a rational balanced polygon cannot be cut into an odd number of triangles of equal areas. First result of this type was obtained by Paul Monsky in 1970. He proved that a square cannot be cut into an odd number of triangles of equal areas. In 2000 Sherman Stein conjectured that the same holds for any balanced polygon. We prove this conjecture in the case when coordinates of all vertices of the cut are rational numbers.

1 Introduction

This paper is motivated by author's attempts to find a new proof of Monsky's theorem, which claims that a square can not be cut¹ into an odd number of triangles of equal areas. This theorem is widely known because of its brilliant proof, which combines techniques from combinatorial topology and arithmetic. The modern explanation of the proof might be found, for example, in [1]. It may seem surprising that no other proofs of this theorem has been found.

In spite of its elegance, the proof of Monsky's theorem has several drawbacks. The main one is that while the statement is, obviously, invariant under the group of affine transformations of the plane, the proof is not. It is based on a construction of a coloring of the plane in such a way that a color of a point depends on the 2-adic valuations of its coordinates. But after applying an affine transformation, the 2-adic valuations of the coordinates change in an uncontrollable way. Another drawback is that this proof seems not to be generalizable to a wider class of polygons, for which the statement holds. The

¹By the phrase *polygon B is cut into triangles* we mean that B is divided into triangles by straight segments. The degenerate situation when a vertex of a triangle lies inside an edge of another one or a polygon itself is allowed.

third drawback is that not of the proof, but of the theorem itself. The statement of Monsky's theorem is rather restricted, it just claims nonexistence of a triangulation with some bizarre property. It seems to be more interesting to find a property of any triangulation, from which Monsky's theorem would follow.

This paper is the result of an attempt to find a proof of Monsky's theorem and its generalizations free of these defects. The generalization we are going to prove is known as the Rational Stein's Conjecture. To give its formulation we need an axillary definition. A polygon is called *balanced* if its edges can be divided into pairs so that in each pair edges are parallel, equal in length and have opposite orientation (the edges are oriented, their orientation comes from the orientation of the boundary). One immediately sees that centrally-symmetric polygons are balanced.

Theorem (The Rational Stein's Conjecture). *It is not possible to cut a balanced polygon into an odd number of triangles having equal areas in a way that all the rectangular coordinates of the triangles are rational numbers.*

Note that a polygon, obtained by unfolding a flat orientable surfaces, is balanced, so it is possible to formulate the statement above for flat surfaces.

Several special cases of this theorem were known before. For centrally-symmetric polygon similar statement was conjectured by Stein and proved by Monsky in 1990 [3]. In [7] Stein proved that a polyomino² of an odd area can not be cut into an odd number of triangles of equal areas, and in 2002 Praton [4] proved the same for an even-area polyomino.

Now we are going to explain our approach to the proof of the Rational Stein's Conjecture. Instead of working with a triangulation (or a cut) of a polygon, we will work with a pair, consisting of a 3-valent graph and a vector function on its edges. The graph will be morally a dual graph of the triangulation, and the function will assign to each edge a vector in the plane, which represents the side, shared by two triangles, corresponding to the vertices of the edge. The function, constructed in this way, will have a property that the sum of the three vectors, corresponding to the three edges with the same terminal vertex, is 0. We will call a 3-valent graph with such a function — a *balanced* graph. For each vertex of a balanced graph one can define its *multiplicity*. It is equal to a 2-adic valuation of a

²By *polyomino* we mean a finite union of squares of area 1 with integer coordinates of vertices.

determinant, constructed from the values of the balancing function on the edges, terminating in the vertex. In original terms it is the 2-adic valuation of the area of the triangle in the triangulation increased by one. The main result of our paper, proved in section 3, is a theorem about balanced graphs. Here is the statement:

Theorem. *Let Γ be a balanced 3-valent graph. Then the number of its vertices, whose multiplicity is minimal among all the vertices of Γ , is even.*

The Rational Stein's Conjecture is an easy corollary of this statement.

Acknowledgments. My gratitude goes to Sergei Tabachnikov for introducing me to the topic and to Nikolai Mnev, without whose guidance and support this article would not have been possible.

2 Balanced graphs and primitive lattices

Our main object of study in the remaining part of the paper will be a pair, consisting of a 3-valent graph and a function, assigning to each edge a pair of 2-adic integers and subject to some conditions. We will call the function "balancing", and a graph with such a function — a "balanced graph".

Firstly, we would like to specify terminology connected with a 3-valent graph. We would prefer to think of it as of undirected talking about cycles, degrees of vertices, etc. Though, in the definition of a balanced graph it is easier to think of it as of directed, simply substituting each undirected edge with a pair of directed edges going in the opposite directions. We hope that this little ambiguity won't lead to any misunderstanding.

We use a standard notation \mathbb{Z}_2 for the ring of 2-adic integers. The 2-adic valuation of a 2-adic integer λ will be denoted by $\nu_2(\lambda)$.

Definition 2.1. *Let Γ be a 3-valent graph. We will call a function B , assigning a pair of 2-adic integers to each of Γ 's directed edges, a balancing function if it satisfies the following two properties:*

- *Let \mathbf{e}^+ and \mathbf{e}^- be two directed edges, corresponding to the same undirected edge \mathbf{e} . Then*

$$B(\mathbf{e}^+) + B(\mathbf{e}^-) = 0.$$

- For any three directed edges $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, sharing the same terminal vertex,

$$B(\mathbf{e}_1) + B(\mathbf{e}_2) + B(\mathbf{e}_3) = 0.$$

We think of its values as of vectors lying in the lattice $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and denote their coordinates by B_x and B_y . The pair, consisting of a graph Γ and a balancing function B is denoted by $\{\Gamma, B\}$.

We will introduce two notions: a *multiplicity* of a vertex and a *lattice* of a vertex. The former is needed to state the main result of our paper, while keeping track of the latter will be the main ingredient of the proof of the main result. From now on we are always working with a balanced graph.

Let a vertex \mathbf{v} be terminal for three edges $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Then we know that $B(\mathbf{e}_1) + B(\mathbf{e}_2) + B(\mathbf{e}_3) = 0$. Therefore, the following definitions make sense:

Definition 2.2. *A multiplicity of a vertex is the 2-adic valuation of the value of the determinant built from any two of the balancing vectors. More concretely,*

$$m(\mathbf{v}) = \nu_2(B_x(\mathbf{e}_1)B_y(\mathbf{e}_2) - B_y(\mathbf{e}_1)B_x(\mathbf{e}_2)).$$

One should bare in mind that the multiplicity of a vertex could be infinite, in this case we think of it as having grater multiplicity, than any one with finite multiplicity. This happens when three balancing vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are collinear. In the proof of the Rational Stein's Conjecture this corresponds to degenerate situation, when a triangle of the triangulation has a vertex on a side of another one.

Definition 2.3. *A lattice of a vertex is a sublattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated over \mathbb{Z}_2 by any two of the balancing vectors. More concretely,*

$$L(\mathbf{v}) = \langle B(\mathbf{e}_1), B(\mathbf{e}_2) \rangle = \mathbb{Z}_2 B(\mathbf{e}_1) + \mathbb{Z}_2 B(\mathbf{e}_2).$$

The lattice of a vertex is of rank 0 or 1 if its multiplicity is infinite and of rank 2 if its multiplicity is finite. From the balancing condition it is clear that in both definitions neither the choice of the pair of vectors nor their order matter. One should bare in mind that the notion of a lattice in a vertex is sharper than that of multiplicity. Indeed, the multiplicity $m(\mathbf{v})$ is just a 2-adic valuation of an index of $L(v)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Now we are ready to formulate the main result.

Theorem 2.4. *Let Γ be a balanced 3-valent graph. Then the number of its vertices, whose multiplicity is minimal among the vertices of Γ , is even.*

For each pair $\{\Gamma, B\}$ we will denote by $M(\{\Gamma, B\})$ the minimal multiplicity of a vertex of Γ .

Our proof will be organized in the following way. We are going to prove theorem 2.4 by induction on M . Firstly, we will prove the base case $M = 0$. In the proof of the induction step we will modify the balancing function B , keeping track of the parity of the number of vertices with multiplicity M . Eventually, we will come to the balancing function B' , whose x and y coordinates on each edge are even 2-adic integers. Dividing the coordinate function by 2, we will construct a balanced graph $\{\Gamma, B''\}$, to which the induction hypothesis can be applied.

Let us call a vector $(u_x, u_y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ primitive if at least one of its coordinates is an odd 2-adic integer. An edge of a balanced graph will be called primitive, if the corresponding vector is primitive. Analogically, we will call a sub-lattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ primitive if it contains a primitive vector. A vertex of a balanced graph will be called primitive if the corresponding lattice is primitive. The main advantage of this notion comes from the following fact:

Lemma 2.5. *Let \mathbf{v} be a vertex of a balanced graph Γ . Then either $m(\mathbf{v}) = 0$ and all the three edges terminating in this vertex are primitive or $m(\mathbf{v}) > 0$ and the number of primitive edges, terminating in this vertex, is even.*

The proof is a simple computation, we will state it after the following corollaries:

Corollary 2.6. *(Base of induction.)*

If $M(\{\Gamma, B\}) = 0$ then Theorem 2.4 holds for $\{\Gamma, B\}$.

Proof. Consider a subgraph \mathcal{P} of Γ , consisting of primitive edges only. By Lemma 2.5, it has only 3-valent and 2-valent vertices. We need to show that the number of the 3-valent vertices is even. Let us denote the number of 2-valent vertices of \mathcal{P} by $\mathcal{V}_2(\mathcal{P})$, 3-valent vertices of \mathcal{P} by $\mathcal{V}_3(\mathcal{P})$ and the number of edges of \mathcal{P} by $\mathcal{E}(\mathcal{P})$. Obviously,

$$2\mathcal{V}_2(\mathcal{P}) + 3\mathcal{V}_3(\mathcal{P}) = 2\mathcal{E}(\mathcal{P}).$$

So, $\mathcal{V}_3(\mathcal{P})$ is even. \square

Corollary 2.7. *If $M(\{\Gamma, B\}) > 0$, then the primitive edges form a system of nonintersecting cycles of Γ .*

Proof. The proof is obvious. \square

These cycles will be called *primitive*. Further, we will work with these cycles separately, modifying the balancing function on each of them. Eventually, we will get rid of all the primitive edges and apply the induction hypothesis. But before that we return to the proof of lemma 2.5.

Proof. Suppose that three edges $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ terminate in the vertex \mathbf{v} . Let $B(\mathbf{e}_i) = (x_i, y_i)$. If for any two of these vectors both coordinates are congruent modulo 2, then $m(\mathbf{v}) > 0$. In this case either they are both primitive and then the third one is not, thanks to the balancing condition, or none of them is primitive. In the latter case both coordinates of the third vector are even 2-adic integers by the same reason.

It remains to analyze the case when all the three vectors are different modulo 2. Applying the balancing condition again, we see that none of them can have two even coordinates. Therefore, these vectors equal $(0, 1)$, $(1, 0)$ and $(1, 1)$ modulo 2. Obviously, in this case

$$m(\mathbf{v}) = \nu_2 \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = 0.$$

\square

In the following we suppose that $M(\Gamma) > 0$ and we are in a position to apply corollary 2.7. From now on we will concentrate on the structure of primitive cycles. The main issue for us will be to understand which lattices can correspond to the vertices of such cycle. Obviously, all these lattices are primitive. The main observation is that the primitive lattices over \mathbb{Z}_2 form some sort of a tree. We will describe their structure in the following two lemmas.

Lemma 2.8. *Let L be a primitive sublattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of multiplicity d . Then for each $0 \leq i \leq d$ there exists exactly one primitive sublattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which contains L and whose multiplicity is equal to i .*

If L contains a primitive vector u with odd first coordinate and i is finite, then the lattice of multiplicity i containing L is generated by the vectors u and $(0, 2^i)$.

If L contains a primitive vector w with odd second coordinate and i is finite, then the lattice of multiplicity i containing L is generated by the vectors w and $(2^i, 0)$.

If L contains a primitive vector w and $i = d$ is infinite, then the lattice of multiplicity i containing L is equal to L . In this case L is generated by w .

Proof. Let M be any primitive lattice between $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and L . Let u be a primitive vector in L . Then it will be a primitive vector in M as well. Without loss of generality suppose that the first coordinate of u is odd. It is well known, and, essentially, a special case of the classification theorem of abelian groups, that there exists a vector $u' \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that $\langle u, u' \rangle$ form a basis of M . But then $\langle u, u' - \frac{u'_x}{u_x} u \rangle$ is a basis as well. Dividing the second vector by invertible 2-adic integer we see that M has a basis $\langle u, (0, 2^i) \rangle$ for some i . Computing the determinant $\det(\langle u, (0, 2^i) \rangle)$ we see that $i = m(M)$. First two statements of the lemma follow from that. The last statement is obvious. \square

In the next lemma we will explain which sub-lattices of minimal index a primitive lattice might have.

Lemma 2.9. *Let L be a primitive lattice of multiplicity $d < \infty$. Then it has exactly three sub-lattices of multiplicity $d+1$. Two of them are primitive (we will denote them L^+ and L^-) and one is not (it will be called L^0). The last one consists of all non-primitive vectors in L . Every primitive vector in L lies either in L^+ or in L^- .*

Proof. It is easy to show that there are only three sub-lattices of multiplicity $d+1$. Leaving the proof to the reader, we will just construct them. Without loss of generality let us suppose that L contains a primitive vector u with

an odd first coordinate. By the previous lemma, L has a basis of the form $\langle u, (0, 2^d) \rangle$. Its primitive sub-lattices of multiplicity $d + 1$ are $\langle u, (0, 2^{d+1}) \rangle$ and $\langle u + (0, 2^d), (0, 2^{d+1}) \rangle$. A non-primitive one is $\langle 2u, (0, 2^d) \rangle$. \square

So, primitive lattices form a 3-valent tree under inclusion with the root $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In this interpretation, multiplicity of a lattice is simply the distance to the root.

Lemma 2.10. *Let \mathbf{v} and \mathbf{w} be two vertices of a balanced graph connected by a primitive edge. If $m(\mathbf{v}) \geq m(\mathbf{w})$, then $L(\mathbf{v}) \subseteq L(\mathbf{w})$.*

Proof. Let us denote the edge, which connects the two vertices, by \mathbf{e} . By lemma 2.8, $L(\mathbf{v})$ has a basis of the form $\langle B(\mathbf{e}), (0, 2^{m(\mathbf{v})}) \rangle$ and $L(\mathbf{w})$ has a basis of the form $\langle B(\mathbf{e}), (0, 2^{m(\mathbf{w})}) \rangle$. From this the statement follows. \square

The following lemma contains information about lattices corresponding to vertices of a primitive cycle, which is essential for our proof. As it has been stated before, we suppose that $M\{\Gamma, B\} > 0$.

Lemma 2.11. *Suppose that vertices $\mathbf{v}_n = \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ form a cycle \mathcal{C} in Γ and are all primitive. If at least one of them is of finite multiplicity, then the following is true:*

1. *Among the lattices $L(\mathbf{v}_0), \dots, L(\mathbf{v}_{n-1})$ there exists one which contains all the others. We will call it maximal and denote by $L(\mathcal{C})$.*
2. *The number of the vertices of the cycle \mathcal{C} , whose lattices are equal to $L(\mathcal{C})$, is even.*
3. *All the vectors, corresponding to the edges, which connect a vertex in the cycle with a vertex not in the cycle, are contained in the lattice $L(\mathcal{C})^0$.*

Proof. 1. Let us form an abstract graph $S(\mathcal{C})$, whose vertices correspond to $\mathbf{v}_0, \dots, \mathbf{v}_{n-1}$. We will use the same symbols to denote the vertices of \mathcal{C} and of $S(\mathcal{C})$. Two vertices \mathbf{a} and \mathbf{b} will be connected by an edge if either $L(\mathbf{a}) \subseteq L(\mathbf{b})$ or $L(\mathbf{b}) \subseteq L(\mathbf{a})$. From the previous lemma we know that \mathbf{v}_i is connected by an edge with \mathbf{v}_{i+1} , so this graph is connected.

Let us take a vertex \mathbf{m} in the cycle, whose lattice $L(\mathbf{m})$ is maximal by inclusion among the lattices $L(\mathbf{v}_i)$. We will show that it contains all other lattices of the cycle. Let us suppose the opposite and take any vertex \mathbf{t} , for which $L(\mathbf{t}) \not\subseteq L(\mathbf{m})$.

Since $S(\mathcal{C})$ is connected, \mathbf{m} and \mathbf{t} can be connected in $S(\mathcal{C})$ by a path of minimal length. If the length of the path is equal to 1, then we come to a contradiction. We know that $L(\mathbf{t}) \not\subseteq L(\mathbf{m})$ by the suggestion about \mathbf{t} and $L(\mathbf{m}) \not\subseteq L(\mathbf{t})$ by the maximality of \mathbf{m} .

We are going to show that the path can be made shorter, which contradicts to its choice. Let us denote its vertices by $\mathbf{w}_0 = \mathbf{m}, \mathbf{w}_1, \dots, \mathbf{w}_1 = \mathbf{t}$. For each j either $L(\mathbf{w}_j) \subseteq L(\mathbf{w}_{j+1})$ or $L(\mathbf{w}_j) \supseteq L(\mathbf{w}_{j+1})$. If for all j the case is the same, then we have either $L(\mathbf{m}) \subseteq L(\mathbf{t})$ or $L(\mathbf{m}) \supseteq L(\mathbf{t})$, none of which is possible. Moreover, by maximality of \mathbf{m} we know that $L(\mathbf{w}_0) \supseteq L(\mathbf{w}_1)$. Therefore, there exists j such that $L(\mathbf{w}_{j-1}) \supseteq L(\mathbf{w}_j) \subseteq L(\mathbf{w}_{j+1})$. Since all these lattices are primitive, it follows from lemma 2.10 that $L(\mathbf{w}_{j-1}) \supseteq L(\mathbf{w}_{j+1})$ or $L(\mathbf{w}_{j-1}) \subseteq L(\mathbf{w}_{j+1})$. So \mathbf{w}_{j-1} and \mathbf{w}_{j+1} are connected in $S(\mathcal{C})$ by an edge and the chosen path is not minimal.

2. Let's denote the edge of Γ connecting \mathbf{v}_i and \mathbf{v}_{i+1} by \mathbf{e}_i . For each edge \mathbf{e}_i we know that $B(\mathbf{e}_i)$ is a primitive vector, $B(\mathbf{e}_i) \in L(\mathcal{C})$. Since at least one vertex of the cycle had finite multiplicity, $L(\mathcal{C})$ has finite multiplicity. First we would like to show that either $B(\mathbf{e}_i) \in L(\mathcal{C})^+$ or $B(\mathbf{e}_i) \in L(\mathcal{C})^-$. If $B(\mathbf{e}_i)$ lies in both $L(\mathcal{C})^+$ and $L(\mathcal{C})^-$, then $L(\mathcal{C})^+ = L(\mathcal{C})^-$ by lemma 2.10 which contradicts lemma 2.9. At the same time, by lemma 2.9, any primitive vector of $L(\mathcal{C})$ is contained in $L(\mathcal{C})^+$ or $L(\mathcal{C})^-$.

Therefore, we can divide the edges of the cycle in two groups: those for which $B(\mathbf{e}_i) \in L(\mathcal{C})^+$ or $L(\mathcal{C})^-$. The evenness of the number of the vertices \mathbf{v} for which $L(\mathbf{v}) = L(\mathcal{C})$ will follow from the following fact: $L(\mathbf{v}_{i+1}) = L(\mathcal{C})$ if and only if $B(\mathbf{e}_i) \in L(\mathcal{C})^+$ and $B(\mathbf{e}_{i+1}) \in L(\mathcal{C})^-$, or $B(\mathbf{e}_i) \in L(\mathcal{C})^-$ and $B(\mathbf{e}_{i+1}) \in L(\mathcal{C})^+$.

The if-part follows from the fact that $B(\mathbf{e}_i)$ and $B(\mathbf{e}_{i+1})$ form a basis of $L(\mathbf{v}_{i+1})$, so if they both are contained in $L(\mathcal{C})^+$ or $L(\mathcal{C})^-$, then the whole lattice $L(\mathbf{v}_{i+1})$ is.

The only-if-part is also easy to show. If $L(\mathbf{v}_{i+1}) \neq L(\mathcal{C})$ then by lemma 2.9 we have $L(\mathbf{v}_{i+1}) \subseteq L(\mathcal{C})^+$ or $L(\mathbf{v}_{i+1}) \subseteq L(\mathcal{C})^-$. In the first case

$B(\mathbf{e}_i)$ and $B(\mathbf{e}_{i+1})$ are contained in $L(\mathcal{C})^+$, in the second case they are contained in $L(\mathcal{C})^-$.

So the vertices, whose lattices are equal to $L(\mathcal{C})$, are exactly those, at which a change of the type of the edge happens. Therefore, there number is even.

3. An edge going from a vertex in the cycle to a vertex not in the cycle is not primitive by corollary 2.7 so it is contained in L^0 by lemma 2.9.

□

Now we can finish the proof of the theorem 2.4.

If $M\{\Gamma, B\} = \infty$, we need to show that the number of vertices of the graph is even. But this is true for any 3-valent graph. So we can suppose that $M\{\Gamma, B\}$ is finite.

We prove the statement for a pair $\{\Gamma, B\}$ by induction on $M\{\Gamma, B\}$. The base follows from corollary 2.6, so we can suppose that $M\{\Gamma, B\} > 0$. By corollary 2.7, the non-primitive edges form a number of nonintersecting cycles. Each primitive cycle either does not contain any vertices of multiplicity M , if its maximal vertex is of greater multiplicity, or contains even number of them, if its maximal vertex is of multiplicity exactly M .

Now we will change B on each edge of each primitive cycle in such a way that all the edges become non-primitive and all the vertices have multiplicity at least $m + 1$. This can be done separately for each primitive cycle. Let us take a cycle \mathcal{C} with vertices $\mathbf{v}_n = \mathbf{v}_0, \dots, \mathbf{v}_{n-1}$ and edges $\mathbf{f}_n = \mathbf{f}_0, \dots, \mathbf{f}_{n-1}$, going out of the cycle. Let $L(\mathcal{C})$ be a maximal lattice of the cycle. Then $m(L(\mathcal{C})) \geq M$ and $m(L(\mathcal{C})^0) \geq M + 1$. All the vectors $B(\mathbf{f}_i)$ are contained in $L(\mathcal{C})^0$ by lemma 2.11. We can modify B on the edges of the cycle, assigning to the edge, which connects \mathbf{v}_i and \mathbf{v}_{i+1} , a vector

$$-\sum_{j=1}^i B(\mathbf{f}_j).$$

It is easy to check that if B was a balancing function, then the modified function will also be balancing. Now for each edge of the cycle the corresponding vector is inside L^0 , so it is non-primitive and all the lattices, corresponding

to the vertices, are contained in L^0 , so they have multiplicity greater than $M + 1$. We can do it for all the primitive cycles consequently and eventually construct a new balancing B' with the desired property.

If all the vertices of $\{\Gamma, B\}$, having multiplicity $M\{\Gamma, B\}$, were primitive, then theorem 1 is proved for $\{\Gamma, B\}$, since we know that in each cycle the number of vertices of multiplicity $M\{\Gamma, B\}$ is even. If not, there exist a non-primitive vertex of multiplicity $M\{\Gamma, B\}$.

Let us consider a function $B'' = \frac{B'}{2}$, which is also balancing by the fact that all the vectors of B' are non-primitive: both their coordinates are even. We know that $M\{\Gamma, B''\} = M\{\Gamma, B\} - 2$, since in $\{\Gamma, B\}$ there was a non-primitive vertex of multiplicity $M\{\Gamma, B\}$. So by the induction hypothesis, the number of vertices in $\{\Gamma, B''\}$ of multiplicity $M\{\Gamma, B\} - 2$ is even. But the number of vertices in $\{\Gamma, B\}$ of multiplicity $M\{\Gamma, B\}$ has the same parity, from which theorem 2.4 follows.

3 Rational Stein's conjecture

We start with recalling a definition of a balanced polygon. A plane polygon with clockwise oriented boundary is called *balanced* if its edges can be divided into pairs so that in each pair edges are parallel, equal in length and have opposite orientation (the edges are oriented, their orientation comes from the orientation of the boundary).

Theorem 3.1 (Rational Stein's Conjecture). *It is not possible to cut a balanced polygon into an odd number of triangles having equal areas in a way that all coordinates of vertices of the triangulation are rational.*

Proof. Let's suppose that such a cut exists and come to a contradiction. Since the statement is invariant under affine transformations of the plane, we can suppose that all coordinates of vertices of the triangulation are integer numbers. Some triangles in the cut can intersect not in the proper way: a vertex of a triangle can lie in the interior of a side of another triangle. By adding additional degenerate triangles of area 0 we can make the triangulation proper. In this modified triangulation there will be an odd number of triangles of equal areas and several triangles of area 0.

From a triangulation of the balanced polygon P we can form a 3-valent graph $\Gamma(P)$ in a natural way. First we take a dual graph of the triangulation. Then we add an extra edge for each pair of the corresponding sides of the boundary of the polygon P . The inclusion of polygon P in the plane determines a balancing function $B(P)$. On each edge \mathbf{e} from the dual triangulation balancing B is defined to be a vector of the common side of the two triangles, corresponding to the ends of \mathbf{e} . For extra edges we can take the corresponding vector of the side of P . Two triangles, corresponding to the ends of the edge have the same vector of the side, because P is balanced. Coordinates of such a vector will be integer numbers, and one can think of them as of 2-adic integers.

Let's suppose that all non-degenerate triangles in the triangulation have the same area S . Multiplicity of a vertex, corresponding to a non-degenerate triangle of $\Gamma(P)$ equals $1 + \nu_2(S)$, while that of a vertex, corresponding to a degenerate triangle is infinite. So, by theorem 2.4 applied to the balanced graph $\{\Gamma(P), B(P)\}$, the number of nondegenerate triangles is even. This leads to the contradiction. \square

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